national accelerator laboratory

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0402

ORBIT AND FOCUSING BUMPS

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The effect of a general orbit bump $\Delta B(z)$ on the closed orbit and the effect of a general focusing bump $\Delta B'(z)$ on the β-function were given by Courant and Snyder (Ann. of Phys. 3, 1-48, 1958, hereafter referred to as C & S.) Here, we put their formulas into easy-to-apply forms and apply them to the main ring. For completeness, we will outline the derivations of the formulas given in C & S.

I. ORBIT BUMP

With an orbit bump $\Delta B(z)$ the orbit equation is

$$\frac{d^2x}{dz^2} + K(z)x = -\frac{\Delta B}{B\rho}. \tag{1}$$

After the Floquet transformation

$$x = \sqrt{\nu\beta}\,u \qquad\qquad dz = \nu\beta\, d\theta \tag{2}$$

we get

$$\frac{d^2u}{d\theta^2} + \nu^2 u = -\frac{\Delta B}{B\rho}(\nu\beta)^{3/2} \equiv F(\theta). \tag{3}$$

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The periodic solution (closed orbit) is

$$u = \frac{1}{4\nu\sin\pi\nu}\left[e^{i\nu(\theta+\pi)}\int_{\theta}^{\theta+2\pi} F(\theta')e^{-i\nu\theta'}d\theta' + e^{-i\nu(\theta+\pi)}\int_{\theta}^{\theta+2\pi} F(\theta')e^{i\nu\theta'}d\theta\right]. \tag{4}$$

[Except for difference in notation this is identical to Eq. (4.7) of C & S]. It is useful to calculate the "invariant" W [Eq. (3.22) of C & S], except now, $W = W(\theta)$ is a function of θ.

$$\begin{aligned} W(\theta) &= \nu\left[u^2 + \frac{1}{\nu^2}\left(\frac{du}{d\theta}\right)^2\right] \\ &= \frac{\nu}{4\nu^2\sin^2\pi\nu}\left(\int_{\theta}^{\theta+2\pi} F(\theta')e^{i\nu\theta'}d\theta'\right)\left(\int_{\theta}^{\theta+2\pi} F(\theta')e^{-i\nu\theta'}d\theta'\right) \\ &= \frac{1}{4\sin^2\pi\nu}\left(\int_{z}^{z+L}\left(\frac{\Delta B}{B\rho}\right)\sqrt{\beta}\; e^{i\phi}dz'\right)\left(\int_{z}^{z+L}\left(\frac{\Delta B}{B\rho}\right)\sqrt{\beta}\; e^{-i\phi}dz'\right) \end{aligned} \tag{5}$$

where

$$\begin{cases} L = \text{orbit length all around} \\ \phi \equiv \displaystyle\int\frac{dz}{\beta} = \text{phase of betatron oscillation.} \end{cases}$$

If the bumps are all localized δ-functions we have

$$\boxed{W(\theta) = \frac{1}{4\sin^2\pi\nu}\left(\sum_n \delta_n\sqrt{\beta_n}\; e^{i\phi_n}\right)\left(\sum_n \delta_n\sqrt{\beta_n}\; e^{-i\phi_n}\right)} \tag{6}$$

where

$$\delta_n \equiv \frac{(\Delta B\ell)_n}{B\rho} = \text{kick angle of the nth bump.}$$

In between two bumps W is constant and the upper-bound of the orbit displacement in that region is given by $\hat{x} = \sqrt{W\beta}$.

Case 1

If we have only one bump ($n = 0$)

$$W = \frac{\beta_o \delta_o^2}{4\sin^2 \pi\nu}.$$

For the main ring if one bending magnet ($\delta_o = 0.0081$) at $\beta_o \cong 90$m is missing we have, since $\sin \pi\nu \cong 1/\sqrt{2}$

$$W \cong \frac{1}{2}\ (90\ \text{m})(0.0081)^2 = 2950\ \text{mm-mrad}.$$

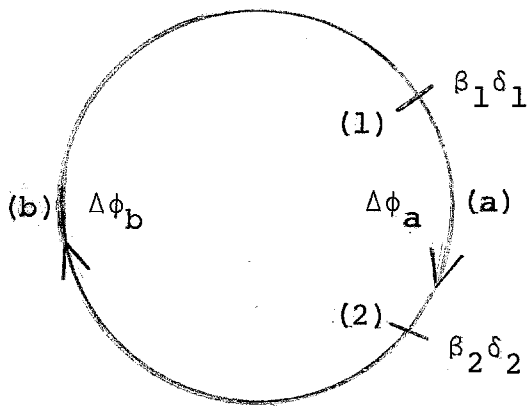
The maximum β is $\beta_{max} \cong 100$ m. This value of W gives for the maximum upper-bound of the closed-orbit displacement

$$\hat{x}_{max} = \sqrt{W\beta_{max}} = 540\ \text{mm}$$

which is, of course, much too large.

Case 2

If we have two bumps $W(\theta)$ has only two values W_a and W_b in region (a) (from bump 1 to bump 2) and region (b) (from bump 2 to bump 1), respectively. They are



$$
\left.\begin{aligned}
W_a &= \frac{1}{4\sin^2\pi\nu}\left(\delta_2\sqrt{\beta_2}\, e^{i\phi_2} + \delta_1\sqrt{\beta_1}\, e^{i\phi_1}\right) \times \\
&\qquad\qquad \left(\delta_2\sqrt{\beta_2}\, e^{-i\phi_2} + \delta_1\sqrt{\beta_1}\, e^{-i\phi_1}\right) \\
&= \frac{1}{4\sin^2\pi\nu}\left(\delta_1^2\beta_1 + 2\delta_1\delta_2\sqrt{\beta_1\beta_2}\cos\Delta\phi_b + \delta_2^2\beta_2\right) \\
W_b &= \frac{1}{4\sin^2\pi\nu}\left(\delta_1^2\beta_1 + 2\delta_1\delta_2\sqrt{\beta_1\beta_2}\cos\Delta\phi_a + \delta_2^2\beta_2\right).
\end{aligned}\right\} \quad (8)
$$

This corresponds to the formulas given in TM-294-Eq.(2). For example, if $\Delta\phi_a = \pi$ and $\delta_1\sqrt{\beta_1} = \delta_2\sqrt{\beta_2}$ we have $W_b = 0$. This is the case of a local orbit bump formed by two magnets π-phase advance apart.

II. FOCUSING BUMP

With a focusing bump $\Delta B'(z)$ the β-deviation equation after the Floquet transformation is

$$
\frac{d^2}{d\theta^2}\left(\frac{\Delta\beta}{\beta}\right) + 4\nu^2\left(\frac{\Delta\beta}{\beta}\right) = -2\frac{\Delta B'}{B\rho}(\nu\beta)^2 \equiv G(\theta). \quad (9)
$$

The periodic solution is

$$
\frac{\Delta\beta}{\beta} = \frac{1}{8\nu\sin 2\pi\nu}\left[e^{2i\nu(\theta+\pi)}\int_\theta^{\theta+2\pi} G(\theta')e^{-2i\nu\theta'}d\theta' + e^{-2i\nu(\theta+\pi)}\int_\theta^{\theta+2\pi} G(\theta')e^{2i\nu\theta'}d\theta'\right]. \quad (10)
$$

(Except for difference in notation this is identical to Eq. (4.50) of C & S). We can define a similar "invariant" $U = U(\theta)$ by

$$U(\theta) = \left(\frac{\Delta\beta}{\beta}\right)^2 + \frac{1}{4\nu^2}\left(\frac{d}{d\theta}\frac{\Delta\beta}{\beta}\right)^2$$

$$= \frac{1}{16\nu^2\sin^2 2\pi\nu}\left(\int_\theta^{\theta+2\pi} G(\theta')e^{2i\nu\theta'}d\theta'\right) x$$

$$\left(\int_\theta^{\theta+2\pi} G(\theta')e^{-2i\nu\theta'}d\theta'\right)$$

$$= \frac{1}{4\sin^2 2\pi\nu}\left(\int_z^{z+L}\left(\frac{\Delta B'}{B\rho}\right)\beta e^{-2i\phi}dz'\right)\left(\int_z^{z+L}\left(\frac{\Delta B'}{B\rho}\right)\beta e^{-2i\phi}dz'\right). \quad (11)$$

If the bumps are localized δ-functions we have

$$\boxed{U(\theta) = \frac{1}{4\sin^2 2\pi\nu}\left(\sum_n \varepsilon_n\beta_n e^{2i\phi_n}\right)\left(\sum_n \varepsilon_n\beta_n e^{-2i\phi_n}\right)} \quad (12)$$

where

$$\varepsilon_n \equiv \frac{(\Delta B'\ell)_n}{B\rho} = \text{focusing "kink" of the nth bump.}$$

In between two bumps U is a constant and the upper-bound of $\Delta\beta$ in that region is $\widehat{\Delta\beta} = \beta\sqrt{U}$.

Case 1

If we have only one bump $(n = 0)$

$$U = \frac{\beta_o^2\varepsilon_o^2}{4\sin^2 2\pi\nu}\ . \quad (13)$$

For the main ring if one focusing quadrupole ($\varepsilon_o = 0.040\ m^{-1}$) at, say, $\beta_{ox} \cong 99m$ and $\beta_{oy} \cong 27$ m is missing, we have, since $\sin 2\pi\nu \cong 1$

$$\begin{cases} U_x \cong \frac{1}{4}\ (99m)^2 (0.040\ m^{-1})^2 = 3.92 \\ U_y \cong \frac{1}{4}\ (27m)^2 (0.040\ m^{-1})^2 = 0.29. \end{cases}$$

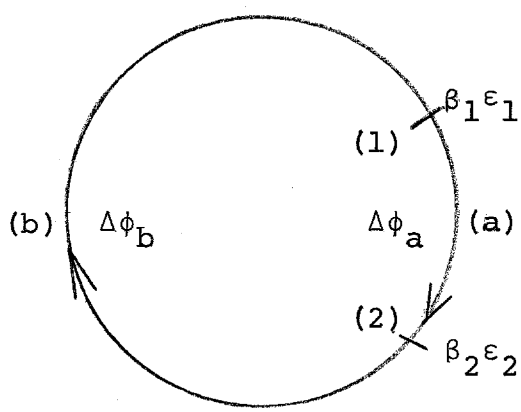
The upper-bounds of the increased β-functions, namely, $B + \widehat{\Delta\beta}$ are, then

$$\begin{cases} (\beta + \widehat{\Delta\beta})_x = (1 + \sqrt{U_x})\beta_x \cong 3.0\ \beta_x \\ (\beta + \widehat{\Delta\beta})_y = (1 + \sqrt{U_y})\beta_y \cong 1.5\ \beta_y. \end{cases} \qquad (14)$$

Since the main ring aperture is rather large these increases in β may well be tolerable.

Case 2

With two bumps $U(\theta)$ has two values U_a and U_b in regions (a) and (b), respectively. In this case we have



$$\begin{cases} U_a = \dfrac{1}{4\sin^2 2\pi\nu}\left(\varepsilon_2\beta_2 e^{2i\phi_2} + \varepsilon_1\beta_1 e^{2i\phi_1}\right) \times \\ \qquad\qquad \left(\varepsilon_2\beta_2 e^{-2i\phi_2} + \varepsilon_1\beta_1 e^{-2i\phi_1}\right) \\ \quad = \dfrac{1}{4\sin^2 2\pi\nu}\left(\varepsilon_1^2\beta_1^2 + 2\varepsilon_1\varepsilon_2\beta_1\beta_2 \cos\Delta\phi_b + \varepsilon_2^2\beta_2^2\right) \\ U_b = \dfrac{1}{4\sin^2 2\pi\nu}\left(\varepsilon_1^2\beta_1^2 + 2\varepsilon_1\varepsilon_2\beta_1\beta_2 \cos\Delta\phi_a + \varepsilon_2^2\beta_2^2\right). \end{cases} \tag{15}$$

Suppose we ask the question whether it is possible to at least partially compensate for a missing quadrupole by turning off a second quadrupole.

U is zero only when $\varepsilon_1\beta_1 = \varepsilon_2\beta_2$ and $\cos\Delta\phi = -1$. For a quadrupole $|\varepsilon|$ has the same value in the x and the y planes. To compensate equally for both planes we should have $\beta_2 = \beta_1$, namely if a focusing quadrupole is missing we should turn off also a focusing quadrupole. Furthermore, since $\Delta\phi_a + \Delta\phi_b = 2\pi\nu \cong 2\pi(20\frac{1}{4})$ to get $\cos\Delta\phi_a$ and $\cos\Delta\phi_b$ equally negative so that U_a and U_b are equally small we should have $\Delta\phi_a = 2\pi(k + \frac{5}{8})$ and $\Delta\phi_b = 2\pi(19 - k + \frac{5}{8})$ with k = integer. Then $\cos\Delta\phi_a = \cos\Delta\phi_b = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$. And we have

$$U_a = U_b \cong \frac{2-\sqrt{2}}{4}\,\varepsilon_1^2\beta_1^2 = 0.146\,\varepsilon_1^2\beta_1^2$$

$$= \begin{cases} 2.30 & \text{x-plane} \\ 0.17 & \text{y-plane} \end{cases}$$

if the first missing quadrupole is a focusing quadrupole. Thus the increase in β is reduced to

$$\begin{cases} (\beta + \widehat{\Delta\beta})_x \cong 2.5\ \beta_x \\ (\beta + \widehat{\Delta\beta})_y \cong 1.4\ \beta_y . \end{cases} \tag{16}$$

Comparing these values with those in Eq. (14) we see that the effect of a missing quadrupole can indeed be partially compensated by turning off another quadrupole, but the amount of compensation is not very large. Here we considered only a compromised compensation in both the x and the y planes and in both regions (a) and (b). It is possible to improve the compensation if for some reason only one of the planes or one of the regions is considered important.